Here is the text from Chapter 1 of my new book: *Equations, the Power and Beautry of Mathematics*, by Clement Falbo

1 WHERE DOES MATHEMATICS COME FROM?

Stone Age tools dating back to about 40,000 years ago were found at a site called the Nwya Devu in Tibet and at earlier dates in the Blombos Cave in South Africa. Also, in many places, drawings and symbols on cave walls depicted human knowledge. There can be no doubt that language and technology had their start in these early prehistoric times. It is safe to speculate that arithmetic, agriculture, and art began to emerge in such places at such times.

Imagine human beings trying to survive by acquiring food and shelter in wild environments. They had to confront major life-threatening problems. Every day they had to think about ways to overcome challenges in a better way today than they had yesterday. The creative human mind is no doubt, the source of our accumulated knowledge, including mathematics, science, literature, art and music that we pass down from one generation to the next.

But what about the stuff we study today in school? We might ask: How did we get to this place? Well, there is plenty of evidence that sophisticated mathematical activities were practiced in civilizations that existed before 3000 BCE. And this happened, independently, in the Eastern World (China), the Near East (Persia), Mesopotamia(Greece), and in the Western World (Europe).

For example, around 500 BCE in China, architects wanted to build a square wall around their city, so they invented the right triangle independently from the Pythagoreans in Greece. Surveyors used formulas to find the areas of the cities enclosed by those walls. Merchants used the abacus to keep track of their purchases, inventories and sales of millet and fish. Warriors wanted to learn the best strategies for winning a battle and gamblers wanted to compute the best odds in betting on the outcomes of wars and games. Thus, we have the beginnings of geometry, computers, matrix algebra and probability in the East. Meanwhile, these same sort of things were happening in the West..

In general, let's say that mathematics comes from human attempts to solve hard problems. Time and time again, new concepts and new branches of mathematics arose when we humans encountered a tough nut to crack. When such a problem was finally solved, it became apparent to the solver and others, even years later, that the same technique would work for totally unrelated problems. Next thing you know, these new mathematical methods would be confronted by some new challenge. Ironically, the new problems could even arise from the previous state of the art itself. This created new and seemingly impossible obstacles, needing new and deeper methods.

Not to belabor this "chicken and egg" theory, let me invite you into my house of mathematics. There is a sense in which mathematics can be defined as a house with a library of collected problems, solved and unsolved In addition, this library contains treatises on formulas and methods that might be worth applying to future problems. It also includes alcoves where visitors can come and sit and try to get insights for attacking problems of their own makings or those found in the library.

They can write up their results and keep them secret, as Carl Friedrich Gauss did in the 1800's with some of his work, ("Few, but ripe." was his motto) or they could put them up on the shelves (publish them) for future visitors. Everyone is welcome to the house of mathematics.

As we enter the house, we realize that much of modern physics, engineering and economics contributed to the mathematics created in the last 3 or 4 centuries. When we study prime numbers we will be discovering beautiful, but hidden, truths that exist in the counting numbers. In our study of abstract algebras, we will have the pleasure of taking stock of powerful techniques and applications that have been discovered in these relatively recent, times. In some of the back rooms, looking at dusty old records of mathematical knowledge dating back beyond thousands of years, we discover the early beginnings of recent events, specifically number theory, geometry and algebra.

Almost all mathematical concepts are derived from the counting numbers, algebra, and geometry. In geometry, we can construct figures, consisting of points, lines and curves, marked on paper, or as done in Athens by the ancient Greek mathematicians, scratched in sand with a stick. Alternatively, geometric entities can exist mentally defined only by a set of assumptions called *axioms.*

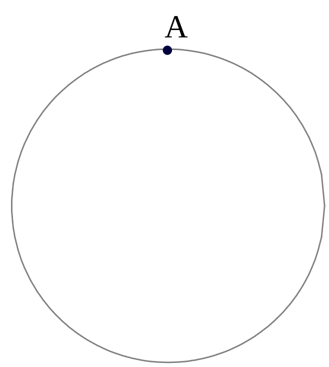

Figure 1.1 Is this a straight line?

In 300 BCE, Euclid brought together three hundred years of Western mathematical knowledge and organized it into thirteen books called, *The Elements*. Euclid's books laid the foundations of plane and solid geometry, number theory, trigonometry, and the beginnings of algebra. Later, around 240 BCE, Archimedes did much more with geometry and numerical computations, applying them to his inventions of engineering tools, war machines, and the solutions of problems requiring calculus, nearly two millennia before the dates usually attributed to the invention of calculus.

The first axiom in Euclidean geometry is "A straight line can be drawn from any point to any point." Scholars over the centuries have been critical of the imprecision of Euclid's treatment of geometry. He failed to meet the modern, more stringent, requirements later applied to evolving standards in mathematics. Consider Figure 1.1. Is this an example of a straight line being drawn from any point to any point? It could be. As a matter of fact, there is no way to settle this question from just this one axiom. The whole geometric system might consist of just this single point, and lines could be defined like this. If Euclid meant (and it is clear that he did) his geometry to contain more than one point, then the axiom is faulty.

In the 18th and 19th centuries, mathematicians re-wrote the axioms and definitions in more precise language; they required this axiom to say that a *straight line can be drawn from any point to any* other *point,* stating clearly that the line is defined by two distinct points, and not by one point alone. Sometimes this axiom is shortened to say "Two points determine a line." What is required for the construction of a line is simply the identification of two points. So that in any geometric figure if we are given that A and B are two (distinct) points then we can justify saying "Construct the line AB through the two given points A and B." The original Euclidean definition of circles and angles also suffered from the same kind of unconscious assumptions that something more was meant than what was said. Never-the-less, if you are willing to forgive such gaffs, Euclid's work was really a substantial achievement, a testament to the greatness of Greek mathematicians.

1.1 Algebra

Actually, the seeds of algebra were sown by Euclid himself when he introduced the idea of a unit length, creating a "metric" (measurement) in geometry in order to add, multiply, subtract and divide lengths, areas and volumes. This type of metric geometry dealt with numbers, and was, eventually, the beginning of algebra, started by Diophantus, in Greece and much further developed in Persia by Al-Khwarizimi around 800 CE. Our word "algebra" came from the title of his book *Al-jabr*, which, apparently, meant it was a book about balancing equations.

1.2 Analytic geometry

For the most part, Euclidean Geometry is *synthetic geometry* because when you solve problems in it you are "synthesizing" (building up) the geometric figures. In 1630, René Descartes introduced the notion of *analytic geometry,* solving problems by "analyzing" them, (breaking them down). He started solving geometric problems by use of algebraic equations and a *coordinate system*, which we call the *Cartesian Coordinates*. In analytic geometry a curve, such as a parabola can be defined by an *equation,* for example the equation, $y = 2x - x^2$, can be represented by the set of all points (x, y), in the coordinate plane where x is any number and y is twice x minus the square of x in the coordinate plane as shown in Figure 1.2. Every point on the graph has coordinates (x, y), where x is any number and $y = 2x - x^2$.

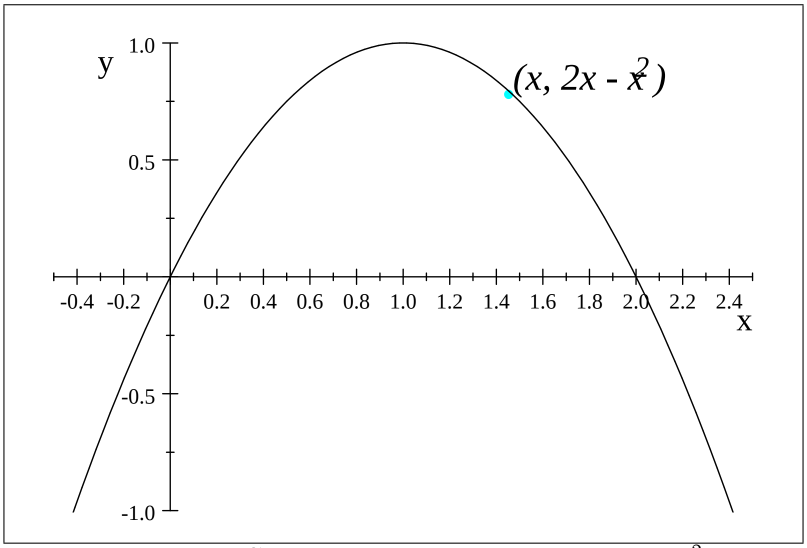

FIGURE 1.2 GRAPH OF THE EQUATION $y = 2x - x^2$.

Analytic geometry lead to a flurry of activities over the 60 years between 1630 and 1690 when other mathematicians such as Pierre de Fermat from France used the coordinate system to start solving problems concerning instantaneous rates of growth, and when Bonaventura Cavalieri from Italy used these coordinates to find areas of geometric regions enclosed by curves. Finally, Isaac Newton from England and, independently, Gotfried von Leibnitz, from Germany harvested all the bits and pieces being produced over these years and amalgamated them into a coherent body of mathematics called the calculus. Newton himself proclaimed in 1675: "If I have seen further, it is by standing on the shoulders of giants."

2 Development of modern mathematics

In Chapter 2, we will see that, at first, algebra was a practical tool used to solve problems in measurements, accounting and construction work. But, astronomy, physics, other sciences and pure mathematical curiosity, itself, moved both algebra and geometry into more advanced forms. Over the centuries, these subjects evolved into the many facets of mathematics today.

Perhaps Euclid's most significant and longest-lasting contribution to mathematics is what really defines formal mathematics. He organized mathematical statements about geometry into a list of things, called the *postulates* or *axioms* which we are willing *to assume* as true (for the sake of argument) versus another list of things, called *propositions, or theorems,* that we *can prove* to be true. Throughout history, any mathematical statement that is claimed to be true in geometry or algebra or any other system, can have that status of being true only if: a) it is one of the assumed postulates of that system *or* b) it is one of the statements that has been proved to be true in that system. But, just what do we mean by "proved to be true"?

2.1 What is a proof?

Put simply, a proof of some assertion, in algebra, geometry (or any other mathematical system) is a sequence of steps that can be used to logically derive that assertion from the axioms of that system. An equally valid proof would be a sequence of steps showing that a *denial* of the assertion leads to a contradiction of the axioms.

In other words, a proof is a method that lets us justify saying that the assertion is *true*–a logical consequence of the axioms. We cannot prove that the axioms themselves are true because we have are already assumed them to be true. Rather than asking the question "are the axioms true?" we need to ask, "are the axioms sufficient for some purpose and do they result in theorems of some substantial consequence?" The study of axioms, themselves in this way is an interesting examination of questions about mathematics and questions about how and why it works. In a sense, it is looking at mathematics from a bird's eye view, and is known as *metamathematics.*

2.2 Rapid progress

In the years between 1630 and 1800 mathematicians concentrated on solving practical problems and it is fair to say that they paid little or no attention to metamathematics. With the discovery, however, of non-euclidean geometry, and imaginary numbers, they recognized the need to firm up the foundations by examining the axioms. This was especially true since many practical problems in gravity, electricity, magnetism, and subatomic forces became more abstract, in physics, and new concepts such as group theory and topology became more abstract in mathematics.

Relativity, space-time continua, various forces in nature, and quantum physics stimulated increasingly new demands on mathematical systems. Around the 1880s mathematicians, and physicists, finally after a lot of introspection, established almost all of the axioms needed to define the real number system.

Even so, one major problem left to be solved was that of the continuity of a function, especially any function that was defined as the limit of an infinite series. The need to fix this problem was "in the air", so to speak, and several mathematicians invented their own axioms to complete the real number line. These various axioms: the *Least Upper Bound Axiom*, the *Nested Sequence Axiom* and others, turned out to be equivalent to each other in the sense that if you assumed any one, you could prove all the others. One of the earliest and most widely accepted such axiom was the one posed by the German mathematician, Richard Dedekind in 1880; his axiom is called the *Dedekind Cut Axiom.*

In any case, the addition of any one of these completeness axioms became the crowning achievement that would insure that the real number line was continuous. Now, the scientific community was sanguine about the most fundamental questions regarding axioms, theorems and logic in mathematics. They could confidently say that "truth" in a system meant "provable" from the assumptions of that system. It is relative and not an absolute.[1]

3 Disturbing questions

At the end of the nineteenth and beginning of the twentieth centuries, the study of sets became central to almost all mathematical ideas. It was the most natural logical consequence in the development of a unified mathematical science. Unfortunately, however, a crises occurred. When the axioms came under closer scrutiny, serious questions arose concerning their logical consistency, their completeness and even their value to humanity. What contradictions arise when we work with sets of numbers or sets of other kinds of elements? Can a set contain itself as an element? We know what we mean by limits, but what do we mean by infinity? Why does it appear that there are two different sizes of infinity; one for the rational numbers and one for the irrational numbers? Are there more infinities? And does *truth* itself exist as an absolute construct? Are all axiom systems equally "good"? And what about Kurt Gödel's proof, upsetting the apple cart in 1930? We will tell you this story in Chapter 12 of this book.

3.1 Practical resolution

Aside from these disturbing questions, what can we properly say about the nature of mathematics and about proofs? Abstractly, mathematics is a collection of non-contradictory logical statements based upon previously laid-out sets of axioms. Even so, we must recognize that mathematics is much more than just that The axioms are based upon real attempts to solve real problems faced by real human beings. For that reason mathematics has become very effective in describing what is going on and what the problems are and what to do about them. Thus, everyday observations and thoughtful guesses find their way into formulating whatever assumptions you want to make as the basis for a mathematical system. Following this path we will see that new mathematical systems are created not only by new axioms, but by new meanings assigned to the operators of *multiplication,* and *addition* as well as to new binary relations. We will see how the axioms of the real number line apply to vector analysis, complex variables, quaternions, and various fields of technology such as computer sciences and engineering.

But let us begin at the beginning. We want to introduce the language of mathematics, specifically mathematical sentences, called *equations*. Believe it or not, the only verb needed in a mathematical sentence is the infinitive form *to be,* and its conjugations: *is*, *are, equals*, *was*, *were*, ..., all denoted by the equal

[1] You will find all of these axioms, including the Dedekind cut axiom in the Chapter 4.

sign, $=$. We also use negations such as: *is not*, *are not*, *will not be*, *not equals*, *is less than*, and *is greater than, ...*denoted by $\neq$, $<$ and $>$.