FROM HERE TO INFINITY

There is an old joke about two Russian generals sitting around thinking about big numbers. They decided to have a contest about who could think of the largest number; the first one says, “Tell me, Comrade, what is the largest number you can think of?” His companion sat for a long while and eventually said, “One.” After hearing that, the first general thought and thought, he wrinkled his brow, made faces and after a pretty long while he gave up, smiled, and said, “You win.”

Actually, no one could ever win in such a guessing contest, because there is no largest number and no matter what number you finally pick, there is always another one larger than that. Such is the nature of infinity.

But even without being able to pin down the elusive word “infinity”, we can still talk about an infinite set in terms of denying the finitude of a set. If there is some integer that is big enough to count all of the objects in a set, then the set is finite, otherwise it is infinite. Here is a definition.

What is finite and what is infinite?

**Definition**: A set *S* is finite if and only if there exists a counting number *n* such that *S* does not contain *n* elements.

For example the set A= {1, 2, 3} contains three elements and is finite because it does not contain 4 elements, nor 5 nor 6, *etc.* That is, you can find a (big enough) counting number *n*, such that the set *A* does not contain that many elements.

 

**Definition**: A set *T* is infinite if and only if it is true that for any positive integer *n*, there are (at least) *n* elements in *T*.

An example of an infinite set is the set of all counting numbers:

 Z = {1, 2, 3, 4, 5, 6, 7 …etc.} 



Another, quite different kind of infinite set on the number line, is the *segment* (1, 3), a piece of the number line, namely all of the numbers between 1 and 3, (not including 1or 3).

 

Every number betweenand is in the segment (1, 3). This includes all fractions, both rational (such as and) and irrational, such asand . The segment *(*1, 3*)* has length 2*,* defined by subtracting the end points: 3-1. In general, if *a* and *b* are any two numbers and *a < b*, then the length of the segment *(a, b)* is the number, *b – a*.

An important question to ask about both finite and infinite sets is: what is the “size” of the set? In 300 BCE, Euclid defined a point as “that which has no part.” By which he meant it has zero length; by which he meant that if you take any line segment, no matter how short, it is still long enough to *cover* that point.

In other words if *P* is any point on the number line then there is a line segment such that *P* is the mid-point of that of the segment. For example if the point is the number *2* on the number line then the segment *(1, 3)* covers *2*. Because *2* is the mid-point of the segment. Not only that, but the line segment *(1.999, 2.001)* also covers *2*. This time, the length of the segment is 0.002. So, we can say the length of the point *2* is less than *0.002*. Or, for that matter you can say the length of the point *2* is less than the tiniest positive number ϵ, you want. Just pick a segment whose mid-point is *2* and whose length is less than ϵ, and you can say the length of *2* is less than ϵ.

If you had a set of two pointsand on the number line then you can prove that the length of the set is smaller than the tiniest number ϵ. Just coverby a segment of length less than ϵ and coverby a segment of length less than ϵ. Therefore the length of the set consisting of two points is ϵ, which is smaller than the amount prescribed. And this is true for any finite set of 3 or 1,000,000 points or more. Furthermore, the infinite set of counting numbers itself, can be covered by a collection of segments whose total length is less that the tiniest positive number you can think of.

Anytime you can say a set, X, of numbers has a length less than any prescribed positive number, then you are saying that X has *measure zero*. Any finite set and all countably infinite sets have measure zero.

**Example**: the set of counting numbers N = {1, 2, 3, 4, 5 …} has measure zero.

**Proof**: Let ϵ be any positive number.

Cover the number  by a segment whose length is less than  (Recall that this means put a segment of length with its midpoint at  on the number line.)

Cover the number 2 by a segment whose length is less than.

Cover the number 3 by a segment whose length is less than 

…

Continue to do this for every positive integer. The sum of the lengths of all these segments is less than ϵ because the infinite series:

 

adds up to ϵ.

Therefore the set of positive integers has measure zero.

# Types of infinities

It turns out that there are countable and uncountable infinities. Let us say what we mean by these two terms.

A Countable infinity

The positive integers {1, 2, 3, 4, 5, 6 …} are the counting numbers and this set is a *countably* infinite set. Any infinite set whose elements can be put into a one-to-one correspondence with the set of counting numbers is a *countably infinite set*. When this happens we say the two sets have the same *cardinalit*y, (the same number of elements.)

The notation we use for the cardinality of the counting numbers is ℵ, “aleph-naught.” 

If you add two countably infinite sets together (that is take their union) you still get a countably infinite set. Example, the set of all even numbers is a countably infinite set because you can match every even number 2, 4, 6,… with the counting numbers 1, 2, 3,…The set of all odd numbers 1, 3, 5,… is also a countably infinite set and it can be matched with the counting numbers. But if you take the union of these two sets, you only need to use the counting numbers once. We summarize this by saying there are just as many even numbers as there are counting numbers. There are just as many odd numbers as there are counting numbers. Each of these sets has cardinality ℵ.

You can use the ordinary counting numbers to count several different countably infinite sets without ever using the same counting number more than once. You can do this by rearranging the counting numbers into several rows.

For example if you have the following four countably infinite sets:

 = {1, 1/4, 1/9, 1/16, 1/25 …} The fractions: 1 divided by squares of integers.

 = {1, 1/8, 1/27, 1/64 …} Fractions made up of 1 divided by cubes of integers.

 Square roots of the positive integers.

 Positive integers raised to the 2/3 power.

And we want to count all four at once with only the counting numbers,

 

We look for a way to re-arrange this one infinitely long string of counting numbers into four infinitely long strings, without repeating any integer as follows:

 

We have woven one infinitely long thread into an infinitely long ribbon, four threads wide. Now we use the first thread in the ribbon, consisting of, to count the infinitely many elements in set A. Then, the second thread, can be used to count the elements in set B, etc. This means we can informally say that

 4×ℵ = ℵ 

This can be done, not just for four but for *any* finite number of infinite sets, you don’t need any more counting numbers than just the positive integers to count all of the elements in all the infinite sets with just the one set of counting numbers. This is saying that the union of any finite number of countably infinite sets, is still only countably infinite

But what if we have more than just a *finite* number of countably infinite sets; that is, what if we have a countably infinite set of countably infinite sets, say ℵ×ℵ, wouldn’t that be more than a countable infinity? Aren’t there more elements in the union of infinitely many infinite sets? Maybe that many aleph-naughts will be uncountable. If not, how can we weave the *one* single thread of counting numbers into an infinitely wide “magic carpet” of infinitely many countable threads?

 To answer this, carry out the following steps (imagine how these steps could be carried out forever.) Start with a 1 in the upper left corner, then under this one place a 2 as the start of a new row. Then go up the first row and place a 3 to the right of the 1 to start a new column. Finish it off with a 4 in row 2 column 2. Notice that the 1 and 4 are on a diagonal from upper left to lower right.

 

Going back to the first column, start a third row with 5 and 6, also going back to the first row start another column with 7 and 8, and finish it off with a 9 in the corner, the third row, third column.

 

Notice that the diagonal now has 1, 4, and 9. In other words, the square of 1 in the first row, first column, the square of 2 in the second row, second column and the square of 3 in the third row third column.

Start a fourth row with 10, 11, 12 and a fourth column with 13, 14, and 15. Finish them off with a 16 in the fourth row fourth column.

 

If you keep on doing this you will keep creating a square with *n* rows and *n* columns with on the diagonal in the nth row, nth column position. This can be carried out forever, creating a matrix with a countable infinity of rows, each containing a countable infinity of elements in each row.

Thei-by-iCounting Tool

We will call this array the “Infinity by Infinity” (i-by-i) Counting Tool.

 

 Now let us use this powerful counting tool to show that the totality of all positive rational numbers is only a countably infinite set.

We are going to be able to match up all of the positive rational number with only one thread of the counting numbers.

The positive rational numbers.

A rational number is any number that can be written as the ratio of two whole numbers. For example, etc. are rational numbers. Any number that cannot be written as the quotient of two whole numbers is not a rational number, and is called an *irrational number*. Some examples of irrational numbers are:  π, etc.

The thing about the rational numbers is that we can list them in a systematic way by taking advantage of the fact that their numerators and denominators are whole numbers. Take any rational number,  and put it into the nth row and mth column of a table. For example you could put the rational number  in the *seventh* row and *sixth* column. Doing this will create a table with infinitely many rows and columns. When we have listed all of the positive rational numbers in such a fashion then we can use the i-by-i Counting Tool to prove that the positive rational numbers has a cardinality of ℵ

 

Such an array will have every positive rational number in it somewhere, also each rational number will occur somewhere in the table more than once for example  will appear in the first row, second column, but it will also appear in the 6th row 3rd column, and in the 10th row fifth column, etc. But, the i-by-i Counting Tool will be able to match all of these rational numbers with the counting numbers. Therefore, there are no more rational numbers than there are counting numbers. So, ℵ× ℵ ℵ Isn’t that amazing?

An uncountable infinity

**The continuum**

 We have just shown that there are only a countable number of rational numbers on the whole positive number line. Are all infinities only countable? Now let us look at segment *(0, 1)*, which is the set of *all* numbers *rational* and *irrational*, between 0 and 1. How many irrational numbers are there? Infinitely many for sure, but does the set of irrational numbers have cardinality ℵ also? Nope. There are more irrationals than rationals. Here is how you can prove it.

 Consider the segment *(0, 1)* on the number line. This segment consists of all of the points, rational and irrational between 0 and 1. The length of this segment is 1. It could be 1 inch, 1 foot, 1 yard, whatever units you use to measure length. Let us say that the segment is a yardstick and its length is one yard.

We have just shown that the *rational* numbers in the segment is only countable; so this set has measure zero. Since there are only rational and irrational numbers in the segment, let us suppose that the *irrationals* are, also, only countable (of measure zero), then the entire segment, which has length 1, is of measure zero, which is impossible. So, the set of irrational numbers must be more than just a countable set.

This means that the cardinality of the real numbers is not ℵ, it is something else. Another name for the set of real numbers is the *continuum*, and the notation for the cardinality of the continuum is **c**, and ***c*** ℵ.

We have a situation that is comparable to the difference between an analog computer and a digital computer. The numbers we input into a digital computer (the usual kind, probably the kind you have) are by strokes tapped on keyboard one at a time. With an analog computer the data are input continuously by a potentiometer, a dial or a sliding lever. The analog is taking in *continuous* data as opposed to *discrete* data. The continuous data are characterized as a continuum. Applications of a continuum are found in light dimming, temperature reading and volume control.

**The Power Set**

But not every different kind of infinity is based on continuous vs discrete counting. A new method invented in the 1880’s uses another method finding sets whose cardinality is greater than ℵ. It might be called a *topological* method because it makes use of the subsets of a set. We use it to find what is called the *power set* of a set. We need a couple of definitions.

**Definition**: **(**Subset) The set **A** is said to be a subset of the set **B** if and only if **A** does not contain any elements that are not in **B**.

The notation saying **A** is a subset of **B** is:

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When **A** is *not* a subset of **B**, then there is an element of **A** that is not in **B**.

**Theorem**: The set empty set, which does not have any elements is a subset of any set.

*Proof*: If there is some set **B** such that is not a subset of **B**, then must contain an element that is not in **B**. But this contradicts the fact that the empty set does not contain any elements.

**Theorem:** The set **B**, itself, is a subset of the set **B**.

*Proof*: The set**B** does not contain any elements that are not in **B**.

**Definition**: (Power set) If **A**is a set then the collection of all the subsets of **A** is called the *power set* of **A**.

**Theorem**: If *m* is the cardinality of a set **S**, then ** is the cardinality of the power set of **S**.

This can easily be proved by mathematical induction for finite sets. Here is an example. If a set has 3 elements, then its power set has 8 (that is: ) elements.

 Example: Given the set , here are its 8 subsets:

 

That is, the cardinality of a power set is always greater than the cardinality of the set. When this concept was applied to finding all of the subsets of the counting numbers, we get a number greater than ℵ. In 1874, a German mathematician, Georg Cantor, recognizing that the cardinality of the power set of the counting numbers, was greater than the cardinality of the original set, and he proposed it as the first uncountable cardinality. He gave this new number a name, calling it ℵ.

So now we had two uncountable infinities: The continuum, **c**, and ℵ. Naturally the question immediately came up; are these two higher infinities the same? Cantor thought that they were and he stated what is known as the *continuum hypothesis*. ***c*** *=* ℵ

**Continuum Hypothesis:** *The cardinality of the continuum is the first uncountable cardinal number.*

Is this true?

In 1939, the Austrian mathematician, Kurt Gödel proved that you could not prove the negation from the axioms of set theory because he could construct a mathematical system in which the continuum hypothesis is true. But in 1993, the American mathematician Paul Cohen proved that you could not prove the continuum hypothesis from the axioms of set theory by constructing an equally valid mathematical system in which it is false. It turns out that it cannot be proved nor disproved.

**Higher Infinities constructed by use of power sets**

Amazingly, there are still even higher cardinalities obtained from the power sets of sets. The next one is: ℵdefined as 2, and so forth.

So, if one of the Russian generals we introduced in the first paragraph had answered “infinity is the largest number I can think of,” the other might say, Tpru! (Whoa!) Which infinity do you mean?